# On Mixing Rank One Infinite Transformations

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D.Ornstein has defined rank one transformations; he has shown that they could be mixing and that they then only commuted with their powers [9]. J.L. King has later proved that mixing rank one transformations had the much stronger property of being MSJ (see [7]). Examples of infinite transformations with MSJ were obtained by J. Aaronson and M. Nadkarni in [1]. J.-P. Thouvenot and the author showed via different approaches that the centralizer of a mixing rank-one *infinite* measure preserving transformation was trivial. Thouvenot used Ornstein's method. In this note the author presents his joining proof based on the technique of [12]. We also consider constructions with algebraic spacers as well as a class of "Sidon constructions" to produce new examples of mixing rank one transformations. In connection with Gordin's question on the existence of homoclinic ergodic actions for a zero entropy system [5],[8] we also discuss Poisson suspensions of some modifications of Sidon rank one constructions.

## 1 Mixing rank one constructions and some applications.

We recall the notion of rank one transformation. Let us consider an infinite (or finite) Lebesgue space $(X, \mu)$. An automorphism (a measure-preserving invertible transformation) $T : X \to X$ is said to be of *rank one*, if there is a sequence $\xi_j$ of measurable partitions of $X$ in the form

$$\xi_j = \{E_j,\ TE_j,\ T^2E_j,\ \ldots, T^{h_j}E_j, \tilde{E}_j\}.$$

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such that $\xi_j$ converges to the partition onto points. The collection

$$E_j, TE_j,\ T^2E_j,\ \dots, T^{h_j}E_j$$

is called Rokhlin's tower. We put

$$U_j = E_j \bigsqcup TE_j \bigsqcup T^2E_j \bigsqcup \dots \bigsqcup T^{h_j}E_j$$

( $\tilde{E}_j$ is the set $X \setminus \bigsqcup_{i=0}^{h_j} T^iE_j$).

**Mixing.** An infinite transformation $T$ is called mixing if for any $\varepsilon > 0$ and sets $A, B$ of finite measures there is $N$ such that for all $|k| > N$

$$\mu(T^kA \cap B) < \varepsilon.$$

To construct examples of mixing rank one infinite transformations one can use the methods by D. Ornstein or T. Adams (see [2],[3]). Below we consider more simple examples.

**Rank one construction** is detrmined by $h_1$ and a sequence $r_j$ of cuttings and a sequence $\bar{s}_j$ of spacers

$$\bar{s}_j = (s_j(1), s_j(2), \dots, s_j(r_j - 1), s_j(r_j)).$$

We recall its definition. Let our $T$ on the step $j$ be assosiated with a collection of disjoint sets (intervals)

$$E_j, TE_jT^2, E_j, \dots, T^{h_j}E_j.$$

We cut $E_j$ into $r_j$ sets (subintervals) of the same measure

$$E_j = E_j^1 \bigsqcup E_j^2 \bigsqcup E_j^3 \bigsqcup \dots \bigsqcup E_j^{r_j},$$

then for all $i = 1, 2, \dots, r_j$ we consider columns

$$E_j^i, TE_j^i, T^2E_j^i, \dots, T^{h_j}E_j^i.$$

Adding $s_j(i)$ spacers we obtain

$$E_j^i, TE_j^iT^2E_j^i, \dots, T^{h_j}E_j^i, T^{h_j+1}E_j^i, T^{h_j+2}E_j^i, \dots, T^{h_j+s_j(i)}E_j^i$$

(the above intervals are disjoint). For all $i < r_j$ we set

$$TT^{h_j+s_j(i)}E_j^i = E_j^{i+1}.$$

Now we obtaine a tower

$$E_{j+1}, TE_{j+1}T^2E_{j+1}, \ldots, T^{h_{j+1}}E_{j+1},$$

where

$$E_{j+1} = E_j^1$$
$$T^{h_{j+1}}E_{j+1} = T^{h_j+s_j(r_j)}E_j^{r_j}$$
$$h_{j+1} + 1 = (h_j+1)r_j + \sum_{i=1}^{r_j} s_j(i).$$

So we have described a general rank one construction.

**Algebraic spacers (a.s.) constructions.** Let $r_j$ be prime, $r_j \to \infty$. We fix and consider generators $q_j$ in the multiplicative groups (associated with the sets $\{1, 2, \ldots, r_j - 1\}$) of the fields $\mathbf{Z}_{r_j}$. For some sequence $\{H_j\}$, $H_j \geq r_j$, we define a spacer sequence

$$s_j(i) = H_j + \{q_j^i\} - \{q_j^{i+1}\}, \quad i = 1, 2, \ldots, r_j - 1,$$

where $H_j$ provides the infinity of our measure space, $s(r_j)$ is arbitrary, $\{q^i\}$ denotes the number from $\{1, 2, \ldots, r_j - 1\}$ that corresponds to $q^i$. This rank one a.s. construction will be mixing. To prove the mixing we must check and apply two properties of our spacers: for $n < r_j$ we have

(1) $-r_j \leq S_j(i,n) := \sum_{k=1}^{n} s_j(i+k) - nH_j \leq r_j, \quad i = 1, 2, \ldots, r_j - n - 1$ (Ornstein's property);

(2) for $i \in \{1, 2, \ldots, r_j - n - 1\}$ all values $S_j(i,n)$ are different (a nice injectivity property).

Indeed, $S_j(i,n) = S_j(m,n)$ implies $q^i - q^{i+n} = q^m - q^{m+n}$, $q^i = q^m$, $i = m$.

For any sequence $n_j < (1-\varepsilon)r_j$ ($\varepsilon > 0$ is fixed) and any positive $f \in L_2$ we get

$$\|P(j,n_j)f\|_{L_2} \to 0, \ j \to \infty,$$

where

$$P(j,n_j) := \frac{1}{r_j - n_j - 1} \sum_{i=1}^{r_j - n_j - 1} T^{S_j(i,n_j)}.$$

Indeed, from (1) and (2) we get

$$\|P(j,n_j)f\|_{L_2} \leq \frac{1}{\varepsilon r_j} \| \sum_{s=-r_j}^{r_j} T^s f\|_{L_2} \quad \to \ 0.$$

Let

$$h_j \leq m_j = n_j(H_j + h_j) + t_j < h_{j+1}, \quad 0 \leq t_j \leq H_j + h_j,$$

then standard rank one method of estimations of mixing via averaging (see [2],[3]) gives

$$\langle T^{m_j}\chi_A,\ \chi_B\rangle \leq \ (\|P(j,n_j)\chi_A\| + \|P(j,n_j+1)\chi_A\| \ + \ \|P(j+1,1)\chi_A\| + \varepsilon)\|\chi_B\|$$

for arbitrary $\varepsilon > 0$ and all large $j$. Thus, for $m \to \infty$ we get

$$\langle T^m\chi_A,\ \chi_B\rangle \to 0.$$

**REMARK.** In fact we can use a.s. constructions (as $H_j$ and $s_j(r_j)$ are sufficiently small) for examples of mixing rank one transformations of a Probability space as well. We follow Ornstein's method but stochastic spacers are replaced now by algebraic ones.

Below we consider mixing constructions only for infinite measure spaces but with an obvious proof of the mixing property.

**Sidon constructions.** Let $r_j \to \infty$ and for each step $j$

$$h_j << s_j(1) << s_j(2) << \ldots << s_j(r_j - 1) << s_j(r_j) \qquad (*).$$

Then for fixed $\xi_{j_0}$-measurable $A, B \in U_{j_0}$ we have

$$\mu(A \cap T^m B) \leq \mu(A)/r_j$$

as $m \in [h_j, h_{j+1}]$, $j > j_0$. Thus, for any $A, B$ of finite measure we get

$$\mu(A \cap T^m B) \to 0.$$

A reader can check that our construction has "Sidon property": only one column in $U_j$ may contain an intersection $U_j \cap T^m U_j$ as $h_{j+1} > m > h_j$. (We recall that Sidon set is a subset $S$ of $\{1, 2, \ldots, N\}$ such that $S \cap S + m$ $(m > 0)$ may contain not more than one point. Its cardinality can be a bit grater than $\sqrt{N}$.) Any Sidon construction (not necessarily satisfied $(*)$) have to be mixing. There are Sidon constructions (with $r_j >> h_j$ but with "minimal" spacers) with a rate of correlations

$$\mu(A \cap T^m A) \leq C\frac{\psi(m)}{\sqrt{m}},$$

where $\psi(m)$ satisfies the condition

$$\frac{\psi(h_{j+1})}{\sqrt{h_{j+1}}} \leq \frac{\psi(m)}{\sqrt{m}}, \quad m \in [h_j+1, h_{j+1}],$$

and $\psi(m) \to \infty$ as slowly as we want (for example, $\psi(m) = \ln\ln(m)$).

Indeed, given $h_j$ we choose $h_{j+1} \sim h_j r_j^2$ such that $\psi(h_{j+1}) \geq \sqrt{h_j}$, then for all $m \in [h_j+1, h_{j+1}]$ we have

$$\mu(A \cap T^m A) \leq \mu(A)/r_j \leq C\frac{\sqrt{h_j}}{\sqrt{h_{j+1}}} = C\frac{\psi(h_{j+1})}{\sqrt{h_{j+1}}}\frac{\sqrt{h_j}}{\psi(h_{j+1})} \leq C\frac{\psi(m)}{\sqrt{m}}.$$

REMARK. In [10] A. Prikhod'ko proposed non-trivial constructions of automorphisms of a Probability space with simple spectrum and the correlation sequence

$$\langle T^m f, f\rangle = O(m^{-1/2+\varepsilon}).$$

We obtained simply the same result for some Sidon constructions due to a spacer freedom in the infinite measure case.

Adding to our Sidon constructions a special (vanishing non-mixing) parts as in [3] we provide simple spectrum for the operator $exp(T)$ and

$$\mu(A \cap T^m B) \leq \varepsilon_j + \mu(A)/r_j,$$

where $\varepsilon_j \to 0$ very very slowly. Following this way we get again one of the results of [3]: there are mixing Poisson suspensions of simple spectrum.

**Homoclinic groups.** The above transformations $T$ can possess ergodic homoclinic groups $H(T)$. Following M.Gordin (see [5],[8]) we recall that

$$H(T) = \{S \in Aut(X,\mu) : \ T^n S T^{-n} \to Id, n \to \infty\}.$$

We see that all $S$ with $\mu(supp(S)) < \infty$ are in $H(T)$. This observation suggest us a construction of a dissipative transformation $S \in H(T)$. Indeed, for $T$ we can build an almost transversal transformation $S$ which is more and more close to the identity on $j$-spacers as $j \to \infty$. If we fix a finite measure set $A \subset U_j$, then $\mu(A\Delta T^n S T^{-n} A) \to 0$ since $T^n S T^{-n}$ will be close to the identity on $A$. It is not so hard to provide $S$ to be dissipative. ( for $n > h_j$ we have $\mu(A\Delta T^n S T^{-n} A) < 2\varepsilon_j + 2\mu(A)/r_j + \delta_j$, where $\varepsilon_j$ as above, and some $\delta_j \to 0$ because our $S$ is more and more close to the identity on $j$-spacers).

Thus we obtain the Poisson suspension $T_*$ with simple spectrum possessing the homoclinic Bernoulli transformation $S_*$ (this is somewhat in contrast with the natural situation in which Bernoulli transformations play the role of $T$). It gives a new solution of Gordin's question with respect to King's one[8]. We note also that *a mixing rank one transformation of a finite measure space has no homoclinic transformations* $S \neq Id$.

M. Gordin also asked the author on an example of a zero entropy transformation with a homoclinic ergodic flow. Now we give a short solution based on Poisson suspensions. Let $\tilde{T}$ a mixing rank one transformation of infinite measure space $(\mathbf{R}^+, \mu)$. We set $X = \mathbf{R} \times \mathbf{R}^+$ and define $T : X \to X$ by the equality

$$T(x, y) = (x, \tilde{T}(y)).$$

It possesses a homoclinic flow. Indeed, let

$$S_t(x, y) = (x + \varphi(y)t, y),$$

where $\varphi > 0$ and $\varphi(y) \to 0$ as $y \to \infty$. The flow $S$ is dissipative and homoclinic: for $f \in L_2(\mathbf{R} \times \mathbf{R}^+, \mu \times \mu)$ we have

$$\|f - T^n S_t T^{-n} f\|_2 \to 0, \quad n \to \infty.$$

The latter is obvious for $f = \chi_{I_1 \times I_2}$ ($I_1, I_2$ are finite intervals), so it is true for all $f \in L_2$. The flow $S_{*t}$ is Bernoulli flow with infinite entropy.

The Poisson suspensions $T_*$ can be of zero entropy. For this we use $\tilde{T}$ with $exp(\tilde{T})$ of simple spectrum. The spectral type of $T_*$ is singular, it coincides with the spectrum of $exp(\tilde{T})$, but the spectrum of $T_*$ has infinite multiplicity.

**REMARK.** In connection with the work [4] let us note that the Poisson suspensions could be a nice source of self-similar flows (with Lebesgue spectrum). Let $\tilde{T}_t$ a flow of infinite measure space $(Y, \mu)$. We set again $X = \mathbf{R} \times Y$ and define $T_t : X \to X$ by the equality

$$T_t(x, y) = (x, \tilde{T}_{e^x t}(y)).$$

Let us consider a flow $G_a$

$$G_a(x, y) = (x + a, y).$$

We get

$$G_a T_t G_{-a} = T_{e^{-a}t},$$
$$G_{*a} T_{*t} G_{*-a} = T_{*e^{-a}t}.$$

We get now that all $S$ with $\mu\times\mu(supp(S)) < \infty$ are in $H(T_t)$. Thus our flows $T_t$ and $T_{*t}$ possess ergodic homoclinic groups. The author is indebted to A.I.Danilenko and E.Roy for arguments (based on [6]) showing that a large class of flows $T_{*t}$ as above could be of zero entropy.

M. Gordin proved that a transformation with an ergodic homoclinic group have to be mixing. We note a bit more:

**Assertion.** *An action with an ergodic homoclinic group is mixing of all orders.*

Thus, Ledrappier's non-multiple-mixing actions (with Lebesgue spectrum) have no ergodic homoclinic groups.

The assertion follows from

**Lemma.** *Let $T_i$ and $T_j$ be sequences of elements of a measure-preserving action on a Probability space, let $S$ satisfy*

*(1) $T_i^{-1}ST_i \to Id$, $T_j^{-1}ST_j \to Id$.*

*If for a measure $\nu$ on $X\times X\times X$ and measurable sets $A, B, C$ it holds*

$$\mu(A\cap T_iB\cap T_jC) \to \nu(A\times B\times C),$$

*then*

$$\nu(SA\times B\times C) = \nu(A\times B\times C).$$

*If in addition $\nu(X\times B\times C) = \mu(B)\mu(C)$ for all $A, B$ and all $S$ satisfied (1) generate together an ergodic action, then $\nu = \mu\times\mu\times\mu$.*

Proof is obvious:

$$\nu(A\times B\times C) = \lim_{i,j}\mu(A\cap T_iB\cap T_jC) = \lim_{i,j}\mu(SA\cap ST_iB\cap ST_jC) =$$

$$= \lim_{i,j}\mu(SA\cap T_iT_i^{-1}ST_iB\cap T_jT_j^{-1}ST_jC) = \nu(SA\times B\times C).$$

If $\nu(X\times B\times C) = \mu(B)\mu(C)$ for all $A, B$, then $\nu$ is a joining of an ergodic action with the identity action on $X\times X$. Thus, $\nu$ have to be the direct product of the projections $\mu$ and $\mu\times\mu$.

## 2 Self-joinings of mixing rank one transformations

A *self-joining* (of order 2) is a $T \times T$-invariant measure $\nu$ on $X \times X$ with marginals equal to $\mu$:

$$\nu(A \times X) = \nu(X \times A) = \mu(A).$$

Off-diagonal measures $\Delta^k$ are defined by the equality

$$\Delta^k(A \times B) = \mu(T^k A \cap B).$$

We say that $T$ has *minimal self-joinings* (*MSJ*) if all its ergodic self-joinings are off-diagonals $\Delta^k$ (a joining is called ergodic if it is not a convex sum of two different joinings).

**THEOREM 1.** *Mixing rank-one infinite transformations have MSJ.*
**COROLLARY.** *A mixing rank-one infinite transformation commutes only with its powers.*

**REMARK.** It follows from corollary and the results of [11] that the Poisson suspensions of the mixing rank one transformations constructed in [3] have trivial centralizer. What is proved here is a partial answer to the question of E. Roy as to whether the Weak Closure Theorem of J. King extends to infinite measure rank one transformations. Theorem 1 in the finite measure case was proved by J. King (see [7],[12]).

Proof of Corollary. Suppose that an automorphism $S$ commutes with $T$. The joining $\Delta_S = (Id \times S)\Delta$ is ergodic: the system $(T \times T, X \times X, \Delta_S)$ is isomorphic to the ergodic system $(T, X, \mu)$. Then for some $i$ we get $(Id \times S)\Delta = (Id \times T^i)\Delta$, so $S = T^i$.

**LEMMA.** *Let $\nu$ be a joining of a rank one transformation $T$. Suppose that $\nu$ is disjoint from all $\Delta^k$. There is a sequence $a_j^k$ such that for all $j$ we have $\sum_k a_j^k \leq 2$ and the inequality*

$$\nu(A \times B) \leq \limsup_j \sum_k a_j^k \Delta^k(A \times B)$$

*holds for all finite measure sets $A, B$.*

Proof of Lemma.

We define sets $C_j^k$ and measures $\Delta_j^k$ :

$$C_j^k = \bigsqcup_{i=0}^{h_j-k} T^i E_j \times T^{i+k} E_j, \quad k \in [0, h_j],$$

$$C_j^k = \bigsqcup_{i=0}^{h_j+k} T^{i-k} E_j \times T^i E_j, \quad k \in [-1, -h_j],$$

$$\Delta_j^k(A \times B) = \Delta_j^k((A \times B) \cap C_j^k), \quad k \in [-h_j, h_j].$$

For $k \in [0, h_j]$ we denote by $N(k, A, B)$ the number of $i$-s r such that the corresponding blocks $T^i E_j \times T^{i+k} E_j$ are in $A \times B$. For $k \in [-1, -h_j]$ we consider the blocks $T^{i-k} E_j \times T^i E_j$. Thus, $N(k, A, B)$ is the number of blocks in $C_j^k \cap (A \times B)$.

Let us define for $k \in [0, h_j]$

$$a_j^k = \frac{\nu(E_j \times T^k E_j)}{\mu(E_j)}$$

and for $k \in [-h_j, -1]$

$$a_j^k = \frac{\nu(T^{-k} E_j \times E_j)}{\mu(E_j)}.$$

Using the invariance $\nu(T^{-k+m} E_j \times T^m E_j) = \nu(T^{-k} E_j \times E_j)$ we get

$$\nu(A \times B) = \mu(E_j) \sum_{|k| \le h_j} a_j^k N(k, A, B) = \sum_{|k| \le h_j} a_j^k \Delta_j^k(A \times B).$$

From this we obtain

$$\sum_{|k| \le h_j} a_j^k \Delta_j^k \to \nu$$

since arbitrary measurable sets $A', B'$ can be approximated by $\xi_j$-measurable ones. We get

$$\nu(A \times B) \le \lim_j \sum_k a_j^k \Delta_j^k(A \times B) \le \limsup_j \sum_k a_j^k \Delta^k(A \times B)$$

and

$$\sum_{|k| \le h_j} a_j^k \le 2.$$

The latter follows from marginal properties of a joining:

$$\mu(E_j)\sum_{k=0}^{h_j} a_j^k = \sum_{k=0}^{k=h_j} \nu(E_j \times T^k E_j) \leq \nu(E_j \times X) = \mu(E_j).$$

Lemma is proved.

To prove Theorem we apply Lemma and a consequence of the mixing: for any $\varepsilon > 0$ there is $M$ such that for all $j$

$$\sum_{|k|>M} a_j^k \Delta^k(A \times B) = \sum_{|k|>M} a_j^k \mu(T^k A \cap B) < 2\varepsilon.$$

If $\nu(A \times B) > 0$, then

$$\limsup_j \sum_k a_j^k \Delta^k(A \times B) > 0,$$

and we find $k'$ such that $a_j^{k'}$ is not vanishing as $j \to \infty$, so

$$\nu \geq a\Delta^{k'}, \quad a > 0,$$

hence, $\nu = \Delta^{k'}$.

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